

## EQUILATERAL TRIANGLE SKEW CONDITION FOR QUASICONFORMALITY

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ABSTRACT. We characterize quasiconformal mappings in terms of the distortion of the vertices of equilateral triangles.

## 1. INTRODUCTION

Since quasiconformal mappings were first studied nearly a century ago, many diverse characterizations have been discovered. These have led to a wide variety of applications in many fields including Teichmüller theory, elliptic PDE's, hyperbolic geometry and complex dynamics. For an overview of these applications and the theory of quasiconformal mappings see [2], [4], and [6]. In this paper we will use the metric definition of quasiconformality to obtain a new formulation for planar quasiconformal mappings.

**Definition 1.** Let  $f : U \rightarrow V$  be a homeomorphism between planar domains. When  $D(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\} \subset U$ , define

$$M(z, r) = \sup\{|f(z) - f(w)| : |z - w| = r\}, \text{ and} \\ m(z, r) = \inf\{|f(z) - f(w)| : |z - w| = r\}.$$

The mapping  $f$  is said to be  $K$ -quasiconformal if

$$H(z) = \limsup_{r \rightarrow 0} \frac{M(z, r)}{m(z, r)} \leq K$$

for a.e.  $z \in U$ , and if  $H(z)$  is bounded in  $U$ .

In his book [6] John Hubbard obtained a new characterization of quasiconformal mappings. Let  $T$  be a closed topological triangle with specified vertices,  $L(T) = \max\{|a - b| : a, b \text{ are vertices of } T\}$  and  $l(T) = \min\{|a - b| : a, b \text{ are vertices of } T\}$ . We define

$$\text{skew}(T) = \frac{L(T)}{l(T)}.$$

Note that  $f(T)$  is also a topological triangle so the expression  $\text{skew}(f(T))$  is defined. Then  $f : U \rightarrow V$  is quasiconformal provided that there exists an increasing homeomorphism  $h : [0, \infty) \rightarrow [0, \infty)$  such that

$$\text{skew}(f(T)) \leq h(\text{skew}(T))$$

for all closed Euclidean triangles  $T \subset U$ . In fact, Hubbard showed that it is sufficient to only consider triangles with skew bounded above by  $\sqrt{7/3}$ . He then asks the question of whether it suffices to only consider equilateral triangles. Progress was made on this problem in a previous paper by Javier Aramayona

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and Peter Haïssinsky [3] in which they show there exists a constant  $\epsilon_0 > 0$  such that if  $\epsilon \in [0, \epsilon_0)$  and

$$\text{skew}(f(T)) \leq 1 + \epsilon$$

for all equilateral triangles  $T$ , then  $f$  is quasiconformal.

Theorem 1 of this paper answers Hubbard's question in the affirmative.

**Theorem 1.** *Let  $U, V$  be two domains in the complex plane  $\mathbb{C}$ , and let  $f : U \rightarrow V$  be an orientation-preserving homeomorphism. For each  $\sigma \geq 1$  there exists  $H(\sigma) \geq 1$  with the following property. If there exists  $\sigma$  such that  $\text{skew}(f(T)) \leq \sigma$  for all equilateral triangles  $T \subset U$ , then, for any  $z \in U$  and any  $r < \text{dist}(z, \mathbb{C} \setminus U)$ , the inequality  $M(z, r) \leq Hm(z, r)$  holds where  $H = H(\sigma)$ . In particular, the map  $f$  is quasiconformal.*

Since quasiconformal maps are differentiable almost everywhere by Mori's theorem [9], we may improve the distortion bounds of quasiconformality. Let  $\text{Skew}(f)$  denote the supremum of  $\text{skew}(f(T))$  over all equilateral triangles contained in  $U$ ; for  $z \in U$  and  $r > 0$ , let  $\text{skew}(f, z, r)$  denote the least upper bound of  $\text{skew}(f(T))$  over all  $T \subset \{w \in U : |z - w| < r\}$ . Set  $\text{skew}(f, z) = \liminf_{r \rightarrow 0} \text{skew}(f, z, r)$  and  $\text{skew}(f) = \|\text{skew}(f, z)\|_\infty$ .

**Corollary 1.** *Let  $U$  be a domain in the complex plane  $\mathbb{C}$ , and let  $f : U \rightarrow f(U)$  be an orientation-preserving homeomorphism with finite  $\text{Skew}(f)$ . If  $\text{skew}(f) \leq \sigma$  then  $f$  is  $K(\sigma)$ -quasiconformal where*

$$K(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

*In particular, if  $\text{skew}(f) = 1$  then  $f$  is a conformal mapping.*

## 2. PROOF OF THE MAIN THEOREM

Throughout the rest of the paper we will use the following notation and conventions:

- (1) We define  $D(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\}$  and let  $C(z, r)$  to be the boundary of  $D(z, r)$ .
- (2) By a curve we mean the image of a not necessarily one-to-one continuous function from a closed interval into  $\mathbb{C}$ .
- (3) All triangles will be closed Euclidean triangles.
- (4) Let  $\mathcal{F}_\sigma$  denote the set of orientation-preserving homeomorphisms of any domain  $U \subseteq \mathbb{C}$  into any domain  $V \subseteq \mathbb{C}$  such that  $\text{skew}(f(T)) \leq \sigma$  for all closed equilateral triangles  $T \subset U$ .

The proof of Theorem 1 relies on the following proposition.

**Proposition 1.** *Let  $U$  be a domain containing  $D(0, 1)$ , let  $f : U \rightarrow \mathbb{C}$  belong to  $\mathcal{F}_\sigma$ , and let  $T$  be the triangle with vertices  $0, 1$  and  $\omega = 1/2 + (\sqrt{3}/2)i$ . Then there exists a disk  $D$  contained in  $f(T)$  such that*

- (1)  $D$  is centered at  $f(p)$  where  $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i \approx 0.5 + 0.29i$ , and
- (2) there exists a constant  $\alpha = \alpha(\sigma)$  such that the radius of  $D$  is at least  $\alpha L(f(T))$ .

We note that if  $f$  is to be quasiconformal, then, certainly, the image  $f(T)$  has to contain a disk of definite size centered at the image of the centroid of the triangle, i.e., the point  $\xi = 1/2 + (\sqrt{3}/6)i$ . Unfortunately, its arithmetic properties make it difficult to relate this point to the vertices of  $T$  using equilateral triangles. The point  $p$  was chosen, because it is both close to the centroid ( $|\xi - p| = \sqrt{3}/(2^9 \cdot 3)$ ), and it is a vertex of a tiling of the plane by equilateral triangles whose vertices include the vertices of  $T$ . Indeed, we have  $p = 1/2 - 85 \cdot 2^{-9} + 85 \cdot 2^{-8}\omega$ , cf. Lemma 1.

We first derive the proof of Theorem 1 from Proposition 1. We will then focus on the proof of the latter.

*Proof of Theorem 1.* Fix  $z \in U$  and  $r > 0$ . If  $D(z, r) \subset U$ , let  $M(z, r) = \max\{|f(z) - f(w)| : w \in C(z, r)\}$  and  $m(z, r) = \min\{|f(z) - f(w)| : w \in C(z, r)\}$ . Denote by  $z_M$  a point in  $C(z, r)$  such that  $|f(z_M) - f(z)| = M(z, r)$ .

Since  $\mathcal{F}_\sigma$  is invariant under pre- and post-composition by affine maps, we may assume that  $z = 0$ ,  $r = 1$ , and  $z_M = 1$ .

Let  $T_1$  be the equilateral triangle with vertices  $0, 1$  and  $\omega$ . Then by Proposition 1, the image of  $T_1$  must contain a disk centered at  $f(p)$  and of radius at least  $\alpha L(f(T_1))$ .

Let us consider the isometry  $A(z) = \overline{z - p}$ . Let  $T_2 = A(T_1)$ . The triangle  $T_2$  is contained in the unit disk, and  $A$  maps  $p$  to  $0$  and  $1$  to  $p$ . Since the other vertices of  $T_2$  lie outside of  $T_1$ , we have  $L(f(T_2)) \geq \alpha L(f(T_1))$ . Moreover, another application of Proposition 1 implies that  $f(T_2)$  contains the disk  $D(f(0), \alpha L(f(T_2)))$ .

Summing up these estimates, we obtain

$$m(0, 1) \geq \alpha L(f(T_2)) \geq \alpha^2 L(f(T_1)) \geq \alpha^2 M(0, 1).$$

□

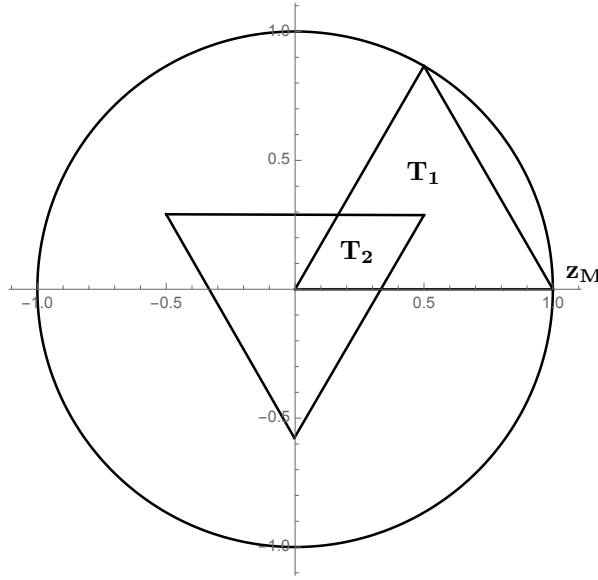


FIGURE 1. Configuration of  $C(0, 1)$ ,  $T_1$  and  $T_2$

### 3. CONSTRUCTION OF CERTAIN TRIANGLES

Proposition 1 is a consequence of the following proposition.

**Proposition 2.** *Let  $U$  be a neighborhood of  $D(0, 1)$ , and let  $f : U \rightarrow \mathbb{C}$  be a homeomorphism onto its image such that  $f \in \mathcal{F}_\sigma$ . Let  $T$  be the closed triangle with vertices  $0, 1$  and  $\omega = 1/2 + (\sqrt{3}/2)i$ . Let  $q = p + 2^{-9}$ . Then there exist points  $t_1, t_2 \in T$  such that the points  $q, t_1, t_2$  form the vertices of an equilateral triangle and the inequalities  $|f(t_j) - f(p)| \leq C\mu$ , and  $|f(p) - f(q)| \geq cL(f(T))$  hold for some constants  $c = c(\sigma)$  and  $C = C(\sigma)$  where  $\mu = \text{dist}(f(p), \mathbb{C} \setminus f(T))$ . We permit the trivial triangle where we have  $t_1 = t_2 = q$ .*

*Proof of Proposition 1 assuming Proposition 2.* If  $t_1 = t_2 = q$ , then we have

$$cL(f(T)) \leq |f(p) - f(q)| \leq C\mu.$$

Otherwise, by the triangle inequality

$$|f(p) - f(q)| - |f(t_1) - f(p)| \leq |f(t_1) - f(q)| \leq \sigma|f(t_1) - f(t_2)| \leq \sigma(|f(t_1) - f(p)| + |f(p) - f(t_2)|)$$

so that by assumption,

$$cL(f(T)) - C\mu \leq 2\sigma C\mu \quad \text{whence} \quad \mu \geq \frac{c}{(2\sigma + 1)C} L(f(T)).$$

□

#### 4. PROOF OF PROPOSITION 2

The idea of our proof of Proposition 2 is to define a curve  $\gamma'$  going through  $p$  such that

- (1) for all  $t \in \gamma'$  we have  $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$ ;
- (2) there are two points  $t_1, t_2 \in \gamma'$ , such that  $q, t_1, t_2$  form the vertices of an equilateral triangle.

The proof of Proposition 2 results from Lemma 1 and Lemma 3.

We first prove the following result.

**Lemma 1.** *Let  $T$  be the closed triangle with vertices  $0, 1$  and  $\omega$ . Let  $p = 1/2 + (85\sqrt{3} \cdot 2^{-9})i$  and  $q = p + 2^{-9}$ . Then  $|f(q) - f(p)| \geq cL(f(T))$  for some positive constant  $c = c(\sigma)$ .*

*Proof.* Let us first consider the tiling of the plane by equilateral triangles with vertices in  $\Lambda = \mathbb{Z} \oplus \omega\mathbb{Z}$ . Define a chain of triangles  $(T_j)_{0 \leq j \leq J}$  as a sequence of triangles with vertices in  $\Lambda$  such that  $T_j \cap T_{j+1}$  is an edge for all  $j$  with  $0 \leq j < J$ . Given two edges  $(v, w)$  and  $(v', w')$ , we may connect them by a chain of minimal length  $n \geq 0$ . A simple induction argument implies

$$|f(v) - f(w)| \leq \sigma^n |f(v') - f(w')|$$

if  $f \in \mathcal{F}_\sigma$  is defined in a neighborhood of the chain.

Let  $T$  be as defined in our hypotheses: it is tiled by  $N = 2^{18}$  triangles of  $2^{-9}\Lambda$ , and  $[p, q]$  is an edge of this tiling. Therefore, for any other edge  $[v, w]$ , it follows that

$$|f(v) - f(w)| \leq \sigma^N |f(p) - f(q)|.$$

But each side of  $T$  is the union of less than  $N$  edges of our tiling, therefore, the triangle inequality implies

$$L(f(T)) \leq N\sigma^N |f(p) - f(q)|.$$

□

We now prove a geometric lemma which will be used in the proof of Lemma 3.

**Lemma 2.** *Let  $|z| \leq 1/8$  and suppose that  $|\theta_\pm - (\pm\pi/3)| \leq 1/8$ . Then the angle  $\theta$  between  $e^{i\theta_+} - z$  and  $e^{i\theta_-} - z$  which crosses the positive real axis belongs to  $(\pi/3, \pi)$ .*

*Proof.* We note that  $\cos \theta_\pm \geq 1/2 - 1/4 > 1/8 \geq |z|$  so that  $\theta$  is less than  $\pi$ .

For the other inequality, we will estimate  $\tan |\arg(e^{i\theta^\pm} - z)|$  to obtain a lower bound of both angles with the horizontal line:

$$\begin{aligned} \tan |\arg(e^{i\theta^\pm} - z)| &\geq \frac{\sqrt{3}/2 - (|z| + 1/8)}{1/2 + (|z| + 1/8)} \geq \frac{\sqrt{3}/2 - 1/4}{1/2 + 1/4} \\ &\geq \frac{2\sqrt{3} - 1}{3} \geq \frac{2}{3} > \tan(\pi/6). \end{aligned}$$

Therefore  $\theta$  is at least  $\pi/3$ . □

Now we demonstrate how to find the curve mentioned above.

**Lemma 3.** *Under the assumptions of Proposition 2, there exists a curve  $\gamma'$  going through  $p$  such that for all  $t \in \gamma'$  we have*

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$$

*and there are two points  $t_1, t_2 \in \gamma'$ , such that  $q, t_1, t_2$  form the vertices of an equilateral triangle. We permit the trivial triangle where we have  $t_1 = t_2 = q$ .*

*Proof.* We will do this in several steps. We first define a curve that will join two points of the boundary of a disk contained in  $T$  (Step 1). To make sure that we will be able to find two points that form an equilateral triangle with  $q$ , we will extend this curve so that it has end points in a slightly larger disk, and is only close to the boundary of the larger disk when it is also close to its end points (Step 2). Then we will use Lemma 2 to find our triangle (Step 3).

Since  $\sqrt{3} \geq 8/5$ , it follows that  $\text{dist}(p, \partial T) = 85\sqrt{3} \cdot 2^{-9} > 1/4 + 2^{-6}$ , so that  $D(p, 1/4 + 2^{-6})$  is contained in the interior of  $T$ .

Throughout the proof, for  $x \in \mathbb{C}$ ,  $R_x$  will denote the rotation centered at  $x$  by  $\pi/3$  radians, defined by  $R_x(z) = x + (z - x)\omega$  and  $\bar{R}_x$  the rotation centered at  $x$  by  $-\pi/3$  radians, defined by  $\bar{R}_x(z) = x + (z - x)\bar{\omega}$ . Recall that we set  $\omega = 1/2 + (\sqrt{3}/2)i$ .

**Step 1: There exists a curve  $\gamma_2$  that satisfies the following:**

- (1)  $\gamma_2 \subset D(p, 1/4)$ ,
- (2)  $\gamma_2$  has end points on  $C(p, 1/4)$  which are exactly  $2\pi/3$  radians apart, and
- (3) for all points  $t \in \gamma_2$  we have  $|f(t) - f(p)| \leq \sigma\mu$ .

Let  $p' \in \partial T$  be such that  $d(f(p), f(p')) = \mu$  and let  $\gamma = f^{-1}([f(p), f(p')])$ . Since  $D(p, 1/4)$  is contained in the interior of  $T$ , we may consider the component  $\gamma_1$  of  $\gamma \cap D(p, 1/4)$  that contains  $p$ , and we denote by  $w \in C(p, 1/4)$  the other end point of  $\gamma_1$ . We take  $w$  to be the first point of  $C(p, 1/4)$  encountered when moving along  $\gamma$  starting from  $p$ .

Now define

$$\gamma_2 = R_p(\gamma_1) \cup \bar{R}_p(\gamma_1).$$

Note that, for any  $s \in \gamma_1$ ,  $R_p(s)$  and  $\bar{R}_p(s)$  are two points in  $\gamma_2$  which make an angle of  $2\pi/3$  seen from  $p$ . Since  $f \in \mathcal{F}_\sigma$ , for all  $t \in \gamma_2$ , we have

$$|f(t) - f(p)| \leq \sigma|f(s) - f(p)| \leq \sigma\mu$$

where  $s \in \gamma_1$  denotes a point such that either  $t = R_p(s)$  or  $t = \bar{R}_p(s)$ .

**Step 2: Let  $a, b$  be the end points of  $\gamma_2$ . There exists a curve  $\gamma_3$  such that**

- (1)  $\gamma_3 \subset D(p, 1/4) \cup D(a, 2^{-6}) \cup D(b, 2^{-6})$ ;
- (2)  $\gamma_3$  has both end points on  $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ ;
- (3) for all points  $t \in \gamma_3$ ,  $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$ .

Let  $D_a = D(a, 2^{-6})$  and  $D_b = D(b, 2^{-6})$ . Let  $\gamma_{2a}$  and  $\gamma_{2b}$  be the components of  $\gamma_2 \cap D_a$  and  $\gamma_2 \cap D_b$  that have end points at  $a$  and  $b$  respectively.

Clearly  $\gamma_{2a}$  also has an end point on the boundary of  $D_a$ . Let  $a'$  denote an end point of  $\gamma_{2a}$  on the boundary of  $D_a$ . Use the tangent line to  $D(p, 1/4)$  at  $a$  to divide  $D_a$  in half, and then divide each half into thirds. So we have divided  $D_a$  into closed sectors of  $\pi/3$  radians with three such sectors lying entirely outside of  $D(p, 1/4)$ . Let  $S_a$  denote the middle sector lying completely outside of  $D(p, 1/4)$ . Then there exists  $n \in \{2, 3\}$  such that when  $\gamma_{2a}$  is rotated  $n\pi/3$  radians in an appropriate direction about  $a$ , the image of  $a'$  under the rotation will lie in  $S_a$ . Let the image of  $\gamma_{2a}$  under this rotation be denoted by  $\gamma_{3a}$ .

Now we will bound the quantity  $|f(t) - f(p)|$  where  $t \in \gamma_{3a}$ . Fix  $t \in \gamma_{3a}$ . Let  $t_0$  be the point on  $\gamma_{2a}$  whose image under the rotation is  $t$ . Without loss of generality we will assume this rotation was clockwise. Let  $t_i$  denote the image of  $t_0$  under a clockwise rotation of  $i\pi/3$  radians where  $i = 1, \dots, n$  ( $t = t_n$ ). Then since  $a, t_{i-1}, t_i$  ( $1 \leq i \leq n$ ) form an equilateral triangle, we have

$$|f(t_i) - f(a)| \leq \sigma|f(t_{i-1}) - f(a)|.$$

Since  $a, t_0 \in \gamma_2$  we have

$$|f(a) - f(t_0)| \leq |f(a) - f(p)| + |f(p) - f(t_0)| \leq 2\sigma\mu$$

and

$$|f(a) - f(p)| \leq \sigma\mu.$$

Thus since  $n$  is at most 3 we have

$$|f(t) - f(p)| \leq |f(a) - f(p)| + |f(a) - f(t)| \leq \sigma\mu + \sigma^n|f(a) - f(t_0)| \leq \sigma\mu(1 + 2\sigma^n) \leq \sigma\mu(1 + 2\sigma^3).$$

Furthermore  $\gamma_{3a}$  must intersect the circle  $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ . This is because  $\gamma_{3a}$  has an end point in  $S_a$  and therefore the distance of the end point of  $\gamma_{3a}$  from  $D(p, 1/4)$  must be at least  $\cos(\pi/6) \cdot 2^{-6} = \sqrt{3} \cdot 2^{-7}$ . This is depicted in figure 2.

We proceed similarly near  $b$  and define a curve  $\gamma_{3b}$  contained in  $D_b$  with end points at  $b$  and at some point on the intersection of the boundary of  $D_b$  and  $S_b$  (defined analogously to  $S_a$ ) such that for all  $t \in \gamma_{3b}$  we have  $|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3)$ ; as above,  $\gamma_{3b}$  intersects the circle  $C(p, 1/4 + \sqrt{3} \cdot 2^{-7})$ .

Let  $\gamma_3$  be the connected component of  $(\gamma_2 \cup \gamma_{3a} \cup \gamma_{3b}) \cap D(p, 1/4 + \sqrt{3} \cdot 2^{-7})$  which includes points in both  $\gamma_{3a}$  and  $\gamma_{3b}$ . Then for all points  $t \in \gamma_3$ ,

$$|f(t) - f(p)| \leq \sigma\mu(1 + 2\sigma^3).$$

The curve  $\gamma'$  in Lemma 3 can be chosen as  $\gamma' = \gamma_3$ .

**Step 3: Let  $q = p + 2^{-9}$ . There exist  $t_1, t_2 \in \gamma_3$  such that  $\{q, t_1, t_2\}$  form an equilateral triangle.**

Let  $D_q$  be the smallest disk centered at  $q$  which contains  $D(p, 1/4)$ . Then  $D_q \subset D(p, 1/4 + \sqrt{3} \cdot 2^{-7})$  since  $|p - q| = 2^{-9} \leq \sqrt{3} \cdot 2^{-8}$ . Let  $\gamma_4$  be the connected component of  $\gamma_3 \cap D_q$  which has end points  $A \in D_a \cap D_q$  and  $B \in D_b \cap D_q$ .

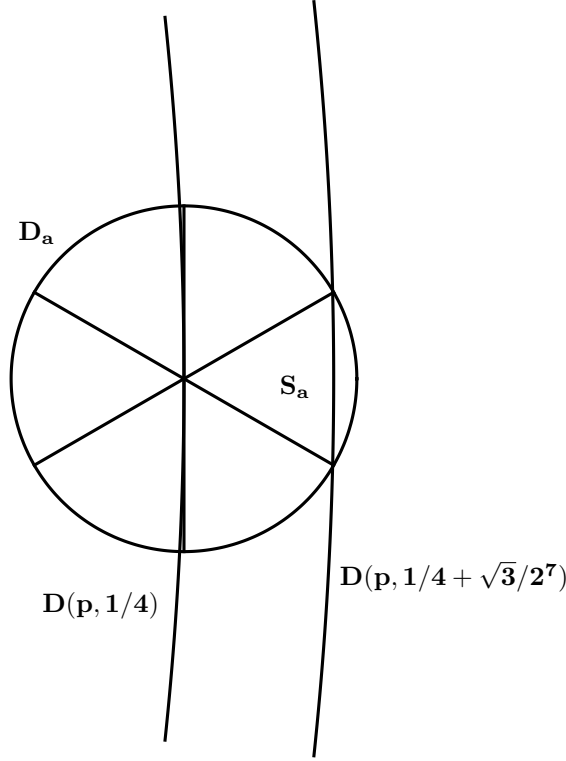


FIGURE 2.

Note that, if we write  $a = p + |a - p|e^{i\theta_a}$  and  $A = p + |A - p|e^{i\theta_A}$ , where  $|\theta_A - \theta_a|$  is chosen to be as small as possible modulo  $2\pi$ , then

$$|\theta_A - \theta_a| \leq 2|a - A|/(1/4) \leq 2 \cdot 2^{-6}/(1/4) \leq 1/8$$

and similarly for  $b$  and  $B$ . Note that  $|\arg(a - p)| + |\arg(b - p)| = 2\pi/3$  and  $|p - q|/(1/4) = 2^{-7} \leq 1/8$ . Therefore, by Lemma 2 applied in  $D(p, 1/4)$ , the angle between  $A - q$  and  $B - q$  lies in  $(\pi/3, \pi)$ . Hence, the images  $A_r$  and  $B_r$  of  $A$  and  $B$  respectively under  $\bar{R}_q$  will separate  $A$  and  $B$  on  $\partial D_q$ . Thus the image  $\bar{R}_q(\gamma_4)$  must intersect  $\gamma_4$ . This gives us our desired equilateral triangle since  $q$ , the intersection point, and the pre-image of the intersection point form an equilateral triangle.

This completes the proof of Lemma 3. □

## 5. PROOF OF COROLLARY 1

We prove Corollary 1 by approximating  $f$  by linear mappings at points where  $f$  is differentiable.

*Proof.* Theorem 1 implies that  $f$  is quasiconformal and hence differentiable at almost every point. Let  $z_0$  be a point of differentiability such that  $\text{skew}(f, z_0) \leq \sigma$ . We will compute the maximum possible value for  $H(z_0)$ . Since  $H(z_0)$  is invariant under Möbius transformations we may compose with translations, a dilation and a rotation to assume that  $z_0 = f(z_0) = 0$ ,  $f_z(z_0) = 1$  and  $f_{\bar{z}}(z_0) = |f_{\bar{z}}(z_0)| < 1$ . Then  $f(z) = z + f_{\bar{z}}(z_0)\bar{z} + \epsilon(z)$  where  $\epsilon(z)/|z|$  tends to 0 as  $z$  tends to  $z_0$ , and thus  $\text{skew}(f, z_0) = \text{skew}(\tilde{f}, z_0)$  where  $\tilde{f}(z) = z + f_{\bar{z}}(z_0)\bar{z}$ .

Note  $|\tilde{f}(a) - \tilde{f}(b)| = |\tilde{f}(a+v) - \tilde{f}(b+v)|$ ,  $|\tilde{f}(a) - \tilde{f}(b)| = |\tilde{f}(\bar{a}) - \tilde{f}(\bar{b})|$  and  $|\tilde{f}(a) - \tilde{f}(b)|/|\tilde{f}(a) - \tilde{f}(c)| = |\tilde{f}(ra) - \tilde{f}(rb)|/|\tilde{f}(ra) - \tilde{f}(rc)|$  for all  $a, b, c, v \in \mathbb{C}$  with  $a \neq c$  and all  $r > 0$ . This implies  $\text{skew}(\tilde{f}(T))$  where  $T$  is an equilateral triangle is invariant under translations, complex conjugation and dilations of  $T$ . Thus for all equilateral triangles  $T$ ,

$$\text{skew}(\tilde{f}(T)) \in \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\}.$$

Indeed, suppose  $T$  has vertices  $A, B$  and  $C$ , and  $\text{skew}(T) = \frac{|\tilde{f}(A) - \tilde{f}(B)|}{|\tilde{f}(A) - \tilde{f}(C)|}$ . First we translate  $A$  to the origin, and then we dilate  $T$  so its side lengths are equal to 1. If  $\overline{AB}$  is  $\pi/3$  radians clockwise from  $\overline{AC}$ , it is clear that our statement is true. Otherwise we take the complex conjugate of  $T$  to change the orientation of  $T$  and then, since  $\tilde{f}$  is invariant under complex conjugation of  $T$ , our claim is true.

Hence we have

$$\text{skew}(\tilde{f}) = \max \left\{ \frac{|\tilde{f}(z) - \tilde{f}(0)|}{|\tilde{f}(ze^{i\pi/3}) - \tilde{f}(0)|} : |z| = 1 \right\} = \max \left\{ \frac{|\tilde{f}(z)|}{|\tilde{f}(ze^{i\pi/3})|} : |z| = 1 \right\}.$$

Let  $\mu = f_{\bar{z}}$ ,  $\nu = \mu + \mu^{-1}$  and  $\beta = e^{i\pi/6}$ . Let  $w \in \mathbb{C}$  with  $|w| = 1$ . We have

$$|\tilde{f}(w)|^2 = |w + \mu\bar{w}|^2 = (w + \mu\bar{w})(\bar{w} + \mu w) = 1 + \mu^2 + \mu(w^2 + \bar{w}^2) = \mu[\nu + (w^2 + \bar{w}^2)].$$

Now we are able to maximize  $|\tilde{f}(\beta w)|/|\tilde{f}(\bar{\beta} w)|$  with respect to  $w$ . Set  $z = w^2$  and  $\alpha = e^{i\pi/3}$ . Since we have assumed  $|w| = 1$ , we can instead maximize

$$\kappa = \left| \frac{\tilde{f}(\beta w)}{\tilde{f}(\bar{\beta} w)} \right|^2 = \frac{\nu + \alpha z + \bar{\alpha}\bar{z}}{\nu + \bar{\alpha}z + \alpha\bar{z}}.$$

We write  $z = e^{ix}$ ,  $x \in \mathbb{R}$ , so that  $z' = iz$ ,  $\bar{z}' = -i\bar{z}$ . We may differentiate  $\kappa$  as a function of  $x$ . It follows that  $\kappa' = 0$  if and only if

$$(\alpha z - \bar{\alpha}\bar{z})(\nu + \bar{\alpha}z + \alpha\bar{z}) - (\nu + \alpha z + \bar{\alpha}\bar{z})(\bar{\alpha}z - \alpha\bar{z}) = 0.$$

Thus

$$\nu(\alpha z - \bar{\alpha}\bar{z} - \bar{\alpha}z + \alpha\bar{z}) = z^2 - \alpha^2 + \bar{\alpha}^2 - \bar{z}^2 - z^2 - \alpha^2 + \bar{\alpha}^2 + \bar{z}^2$$

which is equivalent to

$$\nu(z + \bar{z})(\alpha - \bar{\alpha}) = 2(\bar{\alpha} - \alpha)(\bar{\alpha} + \alpha).$$

Therefore

$$\cos x = -\frac{2}{\nu} \cos \frac{\pi}{3} = -\frac{1}{\nu}.$$

It follows that  $\sin^2 x = 1 - 1/\nu^2$  so  $\kappa' = 0$  for

$$z = \frac{1}{\nu} \left( -1 + i\varepsilon\sqrt{\nu^2 - 1} \right)$$



with  $\varepsilon \in \{\pm 1\}$ . For these values of  $z$ , one gets

$$\kappa = \frac{\nu + 2\operatorname{Re}(\alpha z)}{\nu + 2\operatorname{Re}(\bar{\alpha} z)} = \frac{\nu^2 - 1 - \varepsilon\sqrt{3(\nu^2 - 1)}}{\nu^2 - 1 + \varepsilon\sqrt{3(\nu^2 - 1)}}$$

which is maximal for  $\varepsilon = -1$ . So we obtain

$$\operatorname{skew}(\tilde{f})^2 = \frac{\nu^2 - 1 + \sqrt{3(\nu^2 - 1)}}{\nu^2 - 1 - \sqrt{3(\nu^2 - 1)}} = \frac{\sqrt{(\nu^2 - 1)/3} + 1}{\sqrt{(\nu^2 - 1)/3} - 1}.$$

Let us write  $\tau = \operatorname{skew}(\tilde{f})$  so that

$$\sqrt{(\nu^2 - 1)/3} = \frac{\tau^2 + 1}{\tau^2 - 1}, \quad \nu^2 = 3 \left( \frac{\tau^2 + 1}{\tau^2 - 1} \right)^2 + 1 = \frac{4(\tau^4 + \tau^2 + 1)}{(\tau^2 - 1)^2}$$

and

$$\nu = \frac{2\sqrt{\tau^4 + \tau^2 + 1}}{\tau^2 - 1}.$$

Hence

$$\mu^2 - 2\mu \frac{\sqrt{\tau^4 + \tau^2 + 1}}{\tau^2 - 1} + 1 = 0.$$

We compute the reduced discriminant

$$\Delta' = \frac{\tau^4 + \tau^2 + 1}{(\tau^2 - 1)^2} - 1 = \frac{3\tau^2}{(\tau^2 - 1)^2}$$

and we deduce from  $0 < \mu < 1$  that

$$\mu = \frac{\sqrt{\tau^4 + \tau^2 + 1} - \sqrt{3}\tau}{\tau^2 - 1}.$$

Thus

$$K(\tilde{f}) = \frac{1 + \mu}{1 - \mu} = \frac{\tau^2 - 1 + \sqrt{\tau^4 + \tau^2 + 1} - \sqrt{3}\tau}{\tau^2 - 1 - \sqrt{\tau^4 + \tau^2 + 1} + \sqrt{3}\tau}.$$

By assumption,  $\tau \leq \sigma$  so that

$$K(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1} - \sqrt{3}\sigma}{\sigma^2 - 1 - \sqrt{\sigma^4 + \sigma^2 + 1} + \sqrt{3}\sigma} = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

□

## 6. AN ALTERNATIVE PROOF OF THE QUASICONFORMALITY OF MAPPINGS SATISFYING THE HYPOTHESES OF THEOREM 1

From Proposition 1, there are several ways establishing that a mapping  $f$  satisfying the hypotheses of Theorem 1 satisfies the analytic definition of quasiconformality which is equivalent to Definition 1.

**Definition 2.** We say a homeomorphism  $f : U \rightarrow V$  is absolutely continuous on lines if for every rectangle  $R = \{(x, y) : a < x < b, c < y < d\}$  with  $\bar{R} \subset U$ ,  $f$  is absolutely continuous on a.e. interval  $I_x = \{(x, y) : c < y < d\}$  and a.e. interval  $I_y = \{(x, y) : a < x < b\}$ . A mapping  $f$  is quasiconformal if it is absolutely continuous on lines and there exists  $K \geq 1$  such that

$$\max_{\xi} |\partial_{\xi} f(z)| \leq K \min_{\xi} |\partial_{\xi} f(z)| \text{ a.e.}$$

Proposition 1 tells us that the image of every equilateral triangle,  $T$ , contains a disk with radius proportional to  $L(f(T))$ . In [6, Section 4.5], Hubbard uses this to prove that the map belongs to the Sobolev space  $W_{loc}^{1,2}$  by an approximation argument. We propose another approach which shows directly that the

map satisfies the ACL property. Below we will only give a brief sketch of the basic ideas of the proof. For full details please see Section 5.5 of [1].

*Proof.* First we show  $f$  is absolutely continuous on lines. This part of the proof parallels Pfluger's proof that a mapping satisfying the geometric definition of quasiconformality is absolutely continuous on lines. His proof can be found in [10] and is reproduced in English in [8], p. 162. We fix a rectangle  $R = \{(x, y) : a < x < b, c < y < d\}$  and let  $I_y = \{(x, y) : a < x < b\}$  for  $y$  between  $c$  and  $d$ . Define  $A(y)$  to be the area in  $f(R)$  beneath the image of  $I_y$ . Since  $A$  is an increasing function of  $y$ , it is differentiable almost everywhere. We will show  $f|_{I_y}$  is absolutely continuous for all  $y$  at which  $A$  is differentiable; a similar argument applies to vertical line segments. We select an arbitrary collection  $\{(z_k^*, z_k)\}_{k=1}^n$  of disjoint sub-intervals of  $I_y$  and consider the collection of rectangles  $\{R_k\}_{k=1}^n$  where each  $R_k$  has height  $\delta$  and bottom side on the  $k$ th sub-interval. Then we apply Proposition 1 to a set of non-intersecting equilateral triangles of height  $\delta$  each contained in some  $R_k$ . We are able to conclude

$$\frac{\pi}{\alpha^2} \left( \sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \leq \left( \frac{A(y + \delta) - A(y)}{\delta} \right) \left( \sum_{k=1}^n |z_k^* - z_k| + \delta n \right).$$

Since we chose  $y$  where  $A$  is differentiable, letting  $\delta$  go to 0 gives

$$\frac{\pi}{\alpha^2} \left( \sum_{k=1}^n |f(z_k^*) - f(z_k)| \right)^2 \leq A'(y) \left( \sum_{k=1}^n |z_k^* - z_k| \right).$$

To then conclude that  $f$  is quasiconformal we once again use Proposition 1. Since  $f$  is open and absolutely continuous on lines,  $f$  is differentiable almost everywhere by Gehring and Lehto's theorem [4]. We look at a square  $S_\delta$  with sides parallel to the coordinate axes, side-length  $2\delta$  and center at a point of differentiability. Let  $\gamma$  denote the pre-image under  $f$  of a curve of shortest length contained in  $f(S_\delta)$  with end points on the images of the vertical sides of  $S_\delta$ . We create a chain  $\{T_i\}_{i=1}^4$  of non-intersecting triangles such that each  $T_i$  shares at least one vertex with  $T_{i+1}$ ,  $T_1$  has a vertex on one end point of  $\gamma$ , and  $T_4$  has a vertex on the other end point of  $\gamma$ . By Proposition 1 we are able to conclude that

$$\frac{s_b(f(S_\delta))^2}{m(f(S_\delta))} \leq \frac{4}{(\pi\alpha)^2}$$

where  $m(f(S_\delta))$  denotes the area of the image of  $S_\delta$  and  $s_b(f(S_\delta))$  denotes the length of the shortest curve between the images of the vertical sides of  $S_\delta$  which is contained in  $f(S_\delta)$ . We then may use this inequality to replace the use of Rengel's inequality in Pfluger's proof and conclude

$$\max_{\xi} |\partial_{\xi} f(z)| \leq \frac{4}{(\pi\alpha)^2} \min_{\xi} |\partial_{\xi} f(z)|.$$

Note that once it is known that  $f$  is differentiable almost everywhere, the computations of Corollary 1 also show us that  $f$  is quasiconformal.  $\square$

**Remark 1.** *The key point of both of our proofs given here is contained in Proposition 1, where it is proved that the image of a triangle contains a disk of a definite size, exhibiting a certain length-area estimate. This approach goes back to Pfluger and was pushed forward by Koskela and Rogovin who proved that the ACL property of a homeomorphism  $f$  between open sets of  $\mathbb{R}^n$ ,  $n \geq 2$ , could be established from an  $L^1$ -control of*

$$(1) \quad k_f = \liminf_{r \rightarrow 0} \left( \frac{\text{diam}(f(D(x, r)))^n}{|f(D(x, r))|} \right)^{1/(n-1)}$$

where  $|f(D(x, r))|$  denotes the Lebesgue measure of  $f(D(x, r))$ ; see [7] for details. The authors would like to thank Pekka Koskela for mentioning this similarity.

## 7. APPENDIX BY COLLEEN ACKERMANN: AN ANALOGUE OF THE MAIN THEOREM IN HILBERT SPACES OF DIMENSION AT LEAST THREE

In dimensions three and higher the proof of an analogue of Theorem 1 is surprisingly simpler than the proof of Theorem 1. Furthermore the proof itself gives an elegant bound on  $K(\sigma)$ .

**Theorem 2.** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be finite-dimensional Hilbert spaces with  $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) \geq 3$  and let  $U \subset \mathcal{H}_1, V \subset \mathcal{H}_2$  be domains. Suppose  $f : U \rightarrow V$  is a homeomorphism and that for all closed equilateral triangles  $T \subset U$ ,  $\text{skew}(f(T)) \leq \sigma$ . Then  $f$  is  $\sigma^3$ -quasiconformal when using the metric definition of quasiconformality.*

*Proof.* It suffices to assume  $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^n$  for some  $n \geq 3$ . We will show that  $f$  satisfies the metric definition of quasiconformality.

Fix a point  $p \in U$ , a positive number  $r$  with  $r < \text{dist}(p, \partial U)$  and  $a \in \partial B(p, r)$ . Let  $m \in \partial B(p, r)$  be such that

$$|f(p) - f(m)| = \min_{r=|z-p|} |f(z) - f(p)|.$$

We will prove  $|f(a) - f(p)| \leq \sigma^3 |f(p) - f(m)|$ . Let  $\mathbf{e}_i$  denote the unit vector in the  $i$ th direction. To simplify our calculations we will actually show  $|f(a') - f(p')| \leq K^3 |f(p') - f(m')|$  where  $a', p'$  and  $m'$  are the images of the points  $a, p$  and  $m$  respectively under a sequence of conformal mappings, and where  $f$  is modified accordingly without changing notation. Namely, first apply a translation so that  $p' = 0$ , then a rotation so that  $m' = r\mathbf{e}_1$  and  $a' = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$  and finally a possible reflection so that  $a_2 > 0$ . From now on we will only work in the linear subspace spanned by the first three coordinates which we will identify with  $\mathbb{R}^3$ . More precisely, we will identify  $\mathbf{e}_1$  with the unit vector in the  $x$  direction,  $\mathbf{e}_2$  with the unit vector in the  $y$  direction and another arbitrary coordinate with the  $z$  direction.

**Case 1:** If the smaller angle between  $a'$  and the positive  $x$ -axis is less than or equal to  $2\pi/3$  then set

$$b = \left( r/2, r(1 - a_1)/(2a_2), r\sqrt{3/4 - [(1 - a_1)/(2a_2)]^2} \right).$$

Consider the triangles  $T_1$  with vertices  $p', m'$  and  $b$ , and  $T_2$  with vertices  $p', b$  and  $a'$ .  $T_1$  and  $T_2$  are equilateral triangles which share a common side with endpoints at  $b$  and  $p'$ .

Thus

$$|f(p') - f(a')| \leq \sigma |f(p') - f(b)| \leq \sigma^2 |f(p') - f(m')|.$$

**Case 2:** If the smaller angle between  $a'$  and the  $x$ -axis is not less than or equal to  $2\pi/3$ , consider the equilateral triangle  $T_0$  with vertices  $p', a'$  and  $b'$  where  $b'$  is the image of  $a'$  under a rotation of  $\pi/3$  radians clockwise. The smaller angle between  $b'$  and the  $x$ -axis is less than or equal to  $2\pi/3$ . Thus by Case 1

$$|f(p') - f(b')| \leq \sigma^2 |f(p') - f(m')|.$$

Then since the triangle  $T_0$  has sides with endpoints at  $p'$  and  $a'$ , and  $p'$  and  $b'$  we have

$$|f(p') - f(a')| \leq \sigma |f(p') - f(b')| \leq \sigma^3 |f(p') - f(m')|.$$

□

## REFERENCES

- [1] C. Ackermann, *Quasiconformal mappings on planar surfaces*, Ph.D. Thesis, 2016.

- [2] L. V. Ahlfors, *Lectures on quasiconformal mappings*, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966. MR0200442 (34 #336)
- [3] J. Aramayona and P. Haïssinsky, *A characterisation of plane quasiconformal maps using triangles*, Publ. Mat. **52** (2008), no. 2, 459–471. MR2436734 (2009j:30041)
- [4] K. Astala, T. Iwaniec, and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, vol. 48, Princeton University Press, Princeton, NJ, 2009. MR2472875 (2010j:30040)
- [5] F. W. Gehring and O. Lehto, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Ser. A I No. **272** (1959), 9. MR0124487 (23 #A1799)
- [6] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006. MR2245223 (2008k:30055)
- [7] P. Koskela and S. Rogovin, *Linear dilation and absolute continuity*, Annales Academiae Scientiarum Fennicae Mathematica **30** (2005), 385–392. MR2173371 (2006f:30017)
- [8] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Second, Springer-Verlag, New York-Heidelberg, 1973. Translated from the German by K. W. Lucas, Die Grundlehren der mathematischen Wissenschaften, Band 126. MR0344463 (49 #9202)
- [9] A. Mori, *On quasi-conformality and pseudo-analyticity*, Trans. Amer. Math. Soc. **84** (1957), 56–77. MR0083024
- [10] A. Pfluger, *Über die Äquivalenz der geometrischen und der analytischen Definition quasikonformer Abbildungen*, Comment. Math. Helv. **33** (1959), 23–33. MR0101307

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